

Conformal blocks for AdS_5 singletons

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Abstract

We give a simple derivation of the conformal blocks of the singleton sector of compactifications of IIB string theory on spacetimes of the form $X_5 \times Y_5$ with Y_5 compact, while X_5 has as conformal boundary an arbitrary 4-manifold M_4 . We retain the second-derivative terms in the action for the B, C fields and thus the analysis is not purely topological. The unit-normalized conformal blocks agree exactly with the quantum partition function of the $U(1)$ gauge theory on the conformal boundary. We reproduce the action of the magnetic translation group and the $SL(2, \mathbb{Z})$ S -duality group obtained from the purely topological analysis of Witten. An interesting subtlety in the normalization of the IIB Chern-Simons phase is noted.

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1 Introduction and Conclusion

In AdS compactifications of string theory and M-theory there is a free field sector of the theory known as the singleton sector. In the bulk description these are typically gauge modes, which do not propagate in the interior but do become dynamical on the conformal boundary, thanks to the Chern-Simons terms in the supergravity action. Despite the fact that the singleton sector is “just a free field theory” it has been a source of some confusion. In this paper we give a simple and straightforward derivation of the conformal blocks of the singleton sector for compactifications of type IIB strings on spacetimes of the form $X_5 \times Y_5$, where Y_5 is compact, while X_5 is noncompact with a conformal boundary M_4 . The blocks depend on the topology of M_4 in an interesting way, so we assume M_4 is a general compact 4-manifold. There is an S -duality anomaly if M_4 is not spin. This was observed in [1] and we reproduce the result in section 3.

The singleton sector was first studied by Witten in [1] based on the topological field theory with exponentiated action

$$\exp\left[2\pi i N \int_{X_5} B_2 dC_2\right]. \quad (1.1)$$

Here N is the 5-form flux through X_5 and B_2, C_2 are the supergravity potentials with field strengths H_3, F_3 . Our results are in accord with [1], but in the present paper we retain the second derivative terms in the action. This leads to some differences in the analysis of the conformal blocks. Moreover, as stressed in [2, 3] the Hamiltonian governing the dynamics of the singleton modes is not determined by the Chern-Simons action alone. The method we use determines the Hamiltonian for the singleton modes, and allows us to solve explicitly for the conformal blocks of the singleton sector in terms of Θ -functions. The singleton sector is holographically dual to a free $\mathcal{N} = 4$ supersymmetric theory with gauge group $U(1)$. Our main result is summarized by the Lagrangian for the $U(1)$ gauge boson. The Lagrangian summarizes the coupling of the gauge boson to the harmonic modes of B_2, C_2 at the conformal boundary. It depends on the topological sector $\beta \in H^2(M_4, \mathbb{Z}/N\mathbb{Z})$ and is given by equation (4.23) below. The bulk supergravity interpretation of β is that it is a “Page charge” for the (B_2, C_2) system, much as in [4]. It is quite curious that requiring that the conformal blocks be properly normalized in their natural inner product correctly reproduces the one-loop determinants of the $U(1)$ gauge theory on the boundary. We show this in section 4.3.

The full partition function of the string theory on $X_5 \times Y_5$ will be of the form

$$\sum_{\beta} Z^{\beta} Z_{\beta}^{\text{singleton}} \quad (1.2)$$

Here $Z_{\beta}^{\text{singleton}}$ are the conformal blocks derived in this paper. They are functions of τ and of the harmonic modes of B_2, C_2 on the conformal boundary. The dependence of the partition function on

the remaining boundary values of IIB supergravity fields enter into Z^β . These will be the conformal blocks of a nontrivial, interacting conformally invariant theory. For the case of $AdS_5 \times S^5$ the Z^β are the partition functions of the $SU(N)/\mathbb{Z}_N$, $\mathcal{N} = 4$ SYM theory in the 't Hooft sector β . The wavefunctions Z^β and $Z_\beta^{\text{singleton}}$ transform contragrediently under $SL(2, \mathbb{Z})$ invariance of the IIB supergravity, reflecting the $SL(2, \mathbb{Z})$ invariance of the dual $U(N) = (SU(N) \times U(1))/\mathbb{Z}_N$ $\mathcal{N} = 4$ SYM theory.¹ One should note that *any* compactification holographically dual to a conformal field theory should have a partition function of the form (1.2). For example, in the background $AdS_5 \times T^{1,1}$ discussed in [5] the full gauge group will be

$$\frac{SU(N) \times SU(N) \times U(1)}{\mathbb{Z}_N} \quad (1.3)$$

with the \mathbb{Z}_N diagonally embedded. Similarly in other generalizations such as those discussed in [6, 7, 8, 9] the gauge group behaves analogously and is

$$\frac{SU(N)^k \times U(1)}{\mathbb{Z}_N}$$

with the \mathbb{Z}_N diagonally embedded. Since the $U(1)$ degree of freedom has its origin in the overall center-of-mass degree of freedom in the D-brane picture, constructions such as the warped deformed conifold [10, 11, 12] which add fractional branes will not have such a singleton sector. The reason is that fractional branes are pinned at the orbifold point [13]. This is in accord with the fact that in such geometries the factor of N in (1.1) is logarithmically running, and the topological sector only makes sense for N integral.

The methods used in this paper follow those used in [3] in the analogous case of the AdS_3/CFT_2 correspondence. The same methods can be applied to the AdS_7/CFT_6 correspondence to derive the conformal blocks for the M5 brane of M-theory. (In the latter case the harmonic sector for the C -field does *not* decouple from the massive modes, but may be approximated by a free theory at long distances. The main results were summarized in [4].) The methods of this paper rely on path integrals and are hence not well adapted to the case where $H^*(M_4, \mathbb{Z})$ contains a nontrivial torsion subgroup. However, as pointed out in [1], the theory (including the second derivative terms) is naturally formulated in terms of Cheeger-Simons characters. In appendix A we indicate how our results appear in this formulation, thus extending our results to the case with torsion. We also explain there the tadpole constraint, at the level of integral cohomology.

Finally, it is worth pointing out that the derivation of (1.1) from the 10-dimensional IIB theory is *not* straightforward, contrary to naive expectations. As is well-known, the IIB equations of motion

¹At first sight there is an apparent contradiction with the existence of a baryon vertex. These puzzles, and their resolutions are discussed in [1, 16], [17] p.58, [2] appendix B.

do not follow from a Lorentz-covariant action. However, they can be derived by starting with a Lorentz-covariant action I_{IIB} (given in equation (2.1) below), deriving the equations of motion from $\delta I_{\text{IIB}} = 0$, and then imposing the self-duality of the 5-form on those equations of motion. It is common practice to reduce the action I_{IIB} á la Kaluza-Klein. *This procedure can lead to inconsistent theories*, and in particular leads to (1.1) with N replaced by $N/2$. Such a normalization would lead to an inconsistent quantum theory for N odd. The origin of the trouble is that the action I_{IIB} is not well-defined, because its Chern-Simons term does not carry a proper normalization. This does not, of course, imply any inconsistency in the type IIB supergravity, but it does underscore the fact that the topological phases in the IIB partition functions are very subtle.

2 IIB conventions. Phase of IIB on $X_5 \times Y_5$

The IIB equations of motion can be derived by starting with a Lorentz invariant action on a spin manifold X_{10} and then imposing the self-duality constraint. The action in the Einstein frame is:

$$e^{iI_{\text{IIB}}} = \exp \left[\frac{2\pi i}{g_B^2 \ell_s^8} \int_{X_{10}} \sqrt{-g} \left[\mathcal{R} - \frac{1}{2\tau_2^2} \nabla_\mu \bar{\tau} \nabla^\mu \tau \right] - \frac{i\pi}{2} \int_{X_{10}} R_5 \wedge *R_5 \right. \\ \left. - \frac{i\pi}{g_B \ell_s^4} \int_{X_{10}} \frac{1}{\tau_2} (R_3 + i\tau_2 H_3) \wedge *(R_3 - i\tau_2 H_3) \right] \Phi_B \quad (2.1)$$

where $\tau = C_0 + i\tau_2$ and Φ_B are given below. The Bianchi identities are

$$dR_1 = 0, \quad dH_3 = 0, \quad dR_3 - H_3 \wedge R_1 = 0, \quad dR_5 - H_3 \wedge R_3 = 0. \quad (2.2)$$

Locally they can be solved by

$$H_3 = dB_2, \quad R_1 = dC_0, \quad R_3 = dC_2 - H_3 C_0, \quad R_5 = dC_4 - C_2 \wedge H_3. \quad (2.3)$$

The phase Φ_B is very subtle. Naively this phase is

$$\Phi_B = \exp \left[i\pi \int_{X_{10}} C_4 \wedge H_3 \wedge dC_2 \right].$$

After obtaining equations of motion by varying the action with respect to the potentials B_2, C_0, C_2, C_4 one must impose by hand the additional constraint $R_5 = -*R_5$. Of course, the equations of motion obtained this way do not follow from a Lorentz invariant action. If one ignores this and dimensionally reduces (2.1) anyway, one can obtain an inconsistent quantum theory.

Our considerations are rather general, but for definiteness we note that they apply to Freund-Rubin type backgrounds. The space X_{10} is a product $X_5 \times Y_5$, where Y_5 is a compact manifold.

The metric on X_{10} is a product metric $ds^2 = ds_{X_5}^2 + R^2 ds_{Y_5}^2$. We choose the 5-form flux to be

$$R_5 = \frac{N}{\text{Vol}(Y_5)} [\text{vol}(Y_5) - *_{10} \text{vol}(Y_5)]. \quad (2.4)$$

Here $\text{vol}(Y_5)$ is the volume form on Y_5 , and $\text{Vol}(Y_5)$ is the volume of the compact manifold Y_5 (in our conventions it is dimensionless). Then all equation can be satisfied if we take $\tau = C_0 + i g_B^{-1} = \text{const}$, $F_3 = H_3 = 0$ and

$$\mathcal{R}_{\mu\nu}(X_5) = -\frac{4}{R^2} g_{\mu\nu}(X_5) \quad \text{and} \quad \mathcal{R}_{IJ}(Y_5) = 4 g_{IJ}(Y_5)$$

where

$$R = \ell_s \left[\frac{g_B N}{4 \text{Vol}(Y_5)} \right]^{1/4}. \quad (2.5)$$

One sees that X_5 and Y_5 are negatively and positively curved Einstein manifolds respectively. We will suppose that X_5 has a conformal boundary M_4 . For Y_5 we consider two examples: $Y_5 = S^5$ and $Y_5 = T^{1,1}$ [5].

Now we want to take into account fluctuations of the fields B_2 and C_2 . To this end we need to know the phase of the IIB theory. Although it is not clear how to obtain it directly from the IIB functional (2.1), one can get it indirectly. Fortunately both S^5 and $T^{1,1}$ can be considered as S^1 bundles over \mathbb{CP}^2 and $S^2 \times S^2$ respectively. One can compactify the theory on this S^1 and do T -duality. This untwists the bundle and adds H_3 flux into the IIA background.

2.1 $Y_5 = S^5$

This case was considered in [14]. Any odd dimensional unit sphere S^{2n+1} can be represented as an S^1 bundle over \mathbb{CP}^n . Let $0 \leq \sigma < 1$ be a coordinate on the S^1 . The metric on the unit S^{2n+1} sphere can be written as

$$d\Omega_{2n+1}^2 = (d\sigma + A)^2 + ds_{\mathbb{CP}^n}^2$$

where $ds_{\mathbb{CP}^n}^2$ is the Fubini-Study metric on \mathbb{CP}^n and A is the 1-form on \mathbb{CP}^n . The Ricci tensor of the metric g_{IJ} of the unit sphere S^{2n+1} is $\mathcal{R}_{IJ}(S^{2n+1}) = 2n g_{IJ}$. The metric g_{ij} of \mathbb{CP}^n is normalized such that its Ricci tensor is $\mathcal{R}_{ij}(\mathbb{CP}^n) = (2n+2) g_{ij}$. We also have to require that the curvature F of the $U(1)$ gauge field A must equal $2J$ where J is the Kähler form on \mathbb{CP}^n . The volume form of sphere decomposes as

$$\text{vol}(S^{2n+1}) = \text{vol}(\mathbb{CP}^n) \wedge d\sigma \quad \text{where} \quad \text{vol}(\mathbb{CP}^n) = \frac{1}{n!} J^n. \quad (2.6)$$

The phase for the IIB theory on $X_5 \times S^5$ can be obtained as follows. Consider IIB theory on $X_5 \times S^5$ where S^5 is represented as the Hopf fibration $S^1 \rightarrow S^5 \rightarrow \mathbb{CP}^2$,

$$ds_{10}^2 = ds_{X_5}^2 + R^2 [ds_{\mathbb{CP}^2}^2 + (d\sigma + A^{(R)})^2]$$

where R is the radius of S^5 , and $A^{(R)}$ is a connection form with curvature $\bar{F}_2^{(R)} = 2J$. Then perform the T -duality transformation over S^1 and obtain IIA theory on $X_5 \times \mathbb{C}P^2 \times S^1$ with the nontrivial flux $H_3 = \bar{H}_2 \wedge (d\sigma + A^{(R)})$ [14]. Notice that the T -duality untwists the Hopf fibration and turns it into the direct product. The IIA phase is well defined because it can be obtained by the reduction of the M -theory phase [15]. The 5-form field strength reduces as $R_5 = \bar{R}_5 + \bar{R}_4 \wedge (d\sigma + A^{(R)})$. From Eqs. (2.4) and (2.6) one finds

$$\bar{R}_4 = N \text{vol}(\mathbb{C}P^2). \quad (2.7)$$

Careful matching of the IIA and IIB fluxes on $X_9 \times S^1$ shows that the IIB phase on $X_5 \times S^5$ is

$$\Phi_B = \exp \left[-i\pi \int_{Z_6 \times \mathbb{C}P^2} (\bar{R}_4^2 F_2^{(R)} + 2\bar{R}_4 \bar{R}_3 \bar{H}_3) \right] = \exp \left[-2\pi i N \int_{Z_6} \bar{R}_3 \wedge \bar{H}_3 \right] \quad (2.8)$$

where $\partial Z_6 = X_5$, and we use Eq. (2.7) to obtain the last equality. \bar{R}_3 and \bar{H}_3 comes from the reduction of R_3 and H_3 to X_5 . Notice the “extra” factor of 2 in front of the integral. This justifies (1.1).

2.2 $Y_5 = T^{1,1}$

$T^{1,1}$ can be considered as an S^1 bundle over $S^2 \times S^2$. The metric is [12]

$$ds_{T^{1,1}}^2 = \frac{1}{9} (d\psi + 4\pi A^{(R)})^2 + \frac{1}{6} \left[d\theta_i^2 + \sin^2 \theta_i d\phi_i^2 \right], \quad A^{(R)} = \frac{1}{4\pi} [\cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2] \quad (2.9)$$

where $i = 1, 2$, $0 \leq \theta_i \leq \pi$ and $0 \leq \phi_i < 2\pi$ are coordinates on two S^2 , $0 \leq \psi < 4\pi$ is coordinate on S^1 and $A^{(R)}$ is a connection. The curvature of this connection is

$$F = -\omega_1 - \omega_2 \quad \text{where} \quad \omega_i = \frac{1}{4\pi} \sin \theta_i d\theta_i \wedge d\phi_i.$$

Here ω_i is a generator of $H^2(S^2, \mathbb{Z})$. The metric (2.9) is normalized such that $\mathcal{R}_{IJ}(T^{1,1}) = 4g_{IJ}(T^{1,1})$. Consider IIB theory on $X_5 \times T^{1,1}$:

$$ds_{10}^2 = ds_{X_5}^2 + R^2 ds_{T^{1,1}}^2$$

where R is a “radius” of $T^{1,1}$. Then perform the T -duality transformation over S^1 and obtain IIA theory on $X_5 \times S^2 \times S^2 \times S^1$ with the nontrivial flux $H_3 = \bar{H}_2 \wedge (\frac{1}{4\pi} d\psi + A^{(R)})$. Notice that the T -duality untwists the fibration and turns it into the direct product. The 5-form field strength reduces as $R_5 = \bar{R}_5 + \bar{R}_4 \wedge (\frac{1}{4\pi} d\psi + A^{(R)})$. From Eqs. (2.4) and (2.6) one finds

$$\bar{R}_4 = N \omega_1 \wedge \omega_2. \quad (2.10)$$

Careful matching of the IIA and IIB fluxes on $X_9 \times S^1$ shows that the IIB phase on $X_5 \times S^2 \times S^2 \times S^1$ is

$$\Phi_B = \exp \left[-i\pi \int_{Z_6 \times S^2 \times S^2} (\bar{R}_4^2 F_2^{(R)} + 2\bar{R}_4 \bar{R}_3 \bar{H}_3) \right] = \exp \left[-2\pi i N \int_{Z_6} \bar{R}_3 \wedge \bar{H}_3 \right] \quad (2.11)$$

where $\partial Z_6 = X_5$, and we use Eq. (2.10) to obtain the last equality. \bar{R}_3 and \bar{H}_3 comes from the reduction R_3 and H_3 on X_5 .

Notice that the phase (2.11) of IIB on $X_5 \times T^{1,1}$ and the phase (2.8) of IIB on $X_5 \times S^5$ is the same. In this way we arrive at the topological term (1.1).

2.3 5D Lagrangian for BC fields

The BC part of the Kaluza-Klein reduction of IIB on $X_5 \times Y_5$ is

$$e^{iS_{BC}} = \exp \left[-\frac{i\nu}{2} \int_{X_5} \begin{pmatrix} F_3 & H_3 \end{pmatrix} \mathcal{M}(\tau) \begin{pmatrix} *F_3 \\ *H_3 \end{pmatrix} \right] \Phi_B(B_2, C_2), \quad (2.12a)$$

where $\nu = 4\pi R^5 \text{Vol}(Y_5)/g_B^2 \ell_s^8$ and R is given in (2.5), *locally* $F_3 = dC_2$, $H_3 = dB_2$, $\tau = \text{const}$ and $\mathcal{M}(\tau)$ is correspondingly the complex structure and the metric on the torus

$$\mathcal{M}(\tau) = \frac{1}{\text{Im } \tau} \begin{pmatrix} 1 & -\tau_1 \\ -\tau_1 & |\tau|^2 \end{pmatrix}, \quad \det \mathcal{M}(\tau) = 1. \quad (2.12b)$$

$*$ is Hodge dual with respect to the metric on X_5 .

The phase Φ_B is defined by

$$\Phi_B(B_2, C_2) = \exp \left[2\pi i N \int_{Z_6} H_3 \wedge F_3 \right]. \quad (2.12c)$$

While B_2, C_2 need not be globally well-defined, their fieldstrengths are well-defined. In this formula we have extended H_3, F_3 to a bounding 6-fold.² Suppose we shift $B_2 \rightarrow B_2 + b_2$, $C_2 \rightarrow C_2 + c_2$, where b_2, c_2 are globally well-defined on X_5 . In this case we have the variational formula:

$$\Phi_B(B_2 + b_2, C_2 + c_2) = \Phi_B(B_2, C_2) \exp \left[2\pi i N \int_{X_5} (b_2 F_3 - c_2 H_3) + i\pi N \int_{X_5} (b_2 \wedge dc_2 - c_2 \wedge db_2) \right] \quad (2.13)$$

When X_5 has a nonzero boundary then Φ_B must be considered as a section of a line bundle. In writing the last factor of (2.13) we have chosen a trivialization which is well-adapted to showing the $SL(2, \mathbb{Z})$ invariance. Other choices differ by a total derivative.

²The expression is more properly defined in terms of Cheeger-Simons characters, as indicated in appendix A.

The action (2.12a) is invariant under $SL(2, \mathbb{Z})$ transformations. This duality group acts on the fields as follows

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1; \quad \begin{pmatrix} F'_3 \\ H'_3 \end{pmatrix} = \Lambda \begin{pmatrix} F_3 \\ H_3 \end{pmatrix}, \quad \tau' = \frac{a\tau + b}{c\tau + d}.$$

The classical equations of motion for F_3 and H_3 are

$$\nu d \left[\mathcal{M}(\tau) \begin{pmatrix} *F_3 \\ *H_3 \end{pmatrix} \right] + 2\pi N \begin{pmatrix} -H_3 \\ F_3 \end{pmatrix} = 0. \quad (2.14)$$

This equation implies that F_3 and H_3 are trivial in cohomology.

Near the conformal boundary the manifold X_5 looks like a product $\mathbb{R}_+ \times M_4$. Further we will assume that M_4 is a compact manifold. We are working in a Euclidean formulation of AdS/CFT.

The metric in the vicinity of the conformal boundary is $ds_{X_5}^2 = dr^2/r^2 + r^{-2}ds_{M_4}^2$. The slice $r = 0$ corresponds to the conformal boundary M_4 . More generally we consider metrics of the form

$$ds_{X_5}^2 = d\rho^2 + \Omega^2(\rho) ds_{M_4}^2 \quad (2.15)$$

where $\rho \in \mathbb{R}$. The conformal boundary is located at $\rho = +\infty$. The orientation is $d\rho \wedge d^4x$. We will consider ρ to be a Euclidean time variable $\rho = -it$, and work out the Hamiltonian formalism.

Consider now reduction of the field F_3 (the discussion for H_3 is similar). It reduces as

$$F_3 = \bar{F}(t) + dt \wedge \bar{F}_0(t)$$

where \bar{F} and \bar{F}_0 are 3- and 2-forms on M_4 respectively. The Bianchi identities are

$$d\bar{F} = 0, \quad \partial_0 \bar{F} - d\bar{F}_0 = 0 \quad (2.16)$$

where d is differentiation along M_4 .

At this point we use the Gauss law to conclude that F_3 is topologically trivial, so the global solution of these Bianchi identities is

$$\bar{F}(t) = d\bar{c}(t) \quad \text{and} \quad \bar{F}_0(t) = \partial_0 \bar{c}(t) - d\bar{c}_0(t) \quad (2.17)$$

where $\bar{c}(t), \bar{c}_0(t)$ are globally well-defined.

Now we want to rewrite the action (2.12a) in the new variables. Substituting (2.17) into (2.13) one can write the action as $S_{BC} = \int dt(\mathcal{L}_1 + \mathcal{L}_2)$ where

$$\mathcal{L}_1 = \frac{\nu}{2} \int_{M_4} \begin{pmatrix} \bar{F}_0 & \bar{H}_0 \end{pmatrix} \mathcal{M}(\tau) \begin{pmatrix} *_4 \bar{F}_0 \\ *_4 \bar{H}_0 \end{pmatrix} + \pi N \int_{M_4} (b_0 \wedge d\bar{c} - c_0 \wedge d\bar{b}) + (\bar{b} \wedge \bar{F}_0 - \bar{c} \wedge \bar{H}_0); \quad (2.18a)$$

$$\mathcal{L}_2 = -\frac{\nu}{2\Omega^2} \int_{M_4} \begin{pmatrix} \bar{F} & \bar{H} \end{pmatrix} \mathcal{M}(\tau) *_4 \begin{pmatrix} \bar{F} \\ \bar{H} \end{pmatrix}; \quad (2.18b)$$

and \bar{F}, \bar{H} and \bar{F}_0, \bar{H}_0 are defined in (2.17).

2.4 The Momentum

The momenta are defined by

$$\delta S_{BC} = \int_{\mathbb{R}} dt \int_{M_4} \text{vol}(g) \left[\frac{1}{2} \pi_{\bar{c}}^{ij} \delta(\partial_0 \bar{c}_{ij}) + \frac{1}{2} \pi_{\bar{b}}^{ij} \delta(\partial_0 \bar{b}_{ij}) + \dots \right],$$

where “...” denotes variation of the other fields. Geometrically the momentum is a skewsymmetric bivector field on M_4 . However it is more convenient to regard the momentum as a 2-form on M_4 and define

$$\delta S_{BC} = \int_{\mathbb{R}} dt \int_{M_4} \left[\Pi_{\bar{c}} \wedge \delta(\partial_0 \bar{c}) + \Pi_{\bar{b}} \wedge \delta(\partial_0 \bar{b}) + \dots \right]. \quad (2.19)$$

The relation between these two definitions is $\sqrt{g} \pi_{\bar{c}}^{ij} = \frac{1}{2} \varepsilon^{kl ij} (\Pi_{\bar{c}})_{kl}$. In our conventions $\varepsilon^{kl ij} \in \{\pm 1, 0\}$ and $\varepsilon^{1234} = +1$.

Using (2.18) it is straightforward to show that

$$\begin{pmatrix} \Pi_{\bar{c}} \\ \Pi_{\bar{b}} \end{pmatrix} = \begin{pmatrix} \tilde{\Pi}_{\bar{c}} + \pi N \bar{b} \\ \tilde{\Pi}_{\bar{b}} - \pi N \bar{c} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \tilde{\Pi}_{\bar{c}} \\ \tilde{\Pi}_{\bar{b}} \end{pmatrix} = \nu \mathcal{M}(\tau) *_4 \begin{pmatrix} \bar{F}_0 \\ \bar{H}_0 \end{pmatrix}. \quad (2.20)$$

Here \bar{F}_0, \bar{H}_0 are defined in (2.17). The symplectic form Ω is

$$\Omega = \int_{M_4} \delta \Pi_{\bar{c}} \wedge \delta \bar{c} + \delta \Pi_{\bar{b}} \wedge \delta \bar{b}. \quad (2.21)$$

2.5 The Hamiltonian

The Hamiltonian $\mathcal{H} = \mathcal{H}_e + \mathcal{H}_m$ is given by the Legendre transform

$$\mathcal{L} = \int_{M_4} \left[\Pi_{\bar{c}} \wedge (\partial_0 \bar{c} - d\bar{c}_0) + \Pi_{\bar{b}} \wedge (\partial_0 \bar{b} - d\bar{b}_0) \right] - \mathcal{H}_e - \mathcal{H}_m + \pi N \int_{M_4} \bar{b}_0 \wedge d\bar{c} - \bar{c}_0 \wedge d\bar{b}. \quad (2.22)$$

Straightforward calculation shows that

$$\mathcal{H}_m = \frac{\nu}{2\Omega^2} \int_{M_4} \frac{1}{\tau_2} F_{\tau} *_4 F_{\bar{\tau}}; \quad (2.23a)$$

$$\mathcal{H}_e = \frac{1}{2\nu\tau_2} \int_{M_4} [\tilde{\Pi}_{\tau} *_4 \tilde{\Pi}_{\bar{\tau}} + \tilde{\Pi}_{\bar{\tau}} *_4 \tilde{\Pi}_{\tau}] \quad (2.23b)$$

where

$$\tilde{\Pi}_{\tau} = \tilde{\Pi}_{\bar{b}} + \tau \tilde{\Pi}_{\bar{c}} \quad \text{and} \quad \tilde{\Pi}_{\bar{\tau}} = \tilde{\Pi}_{\bar{b}} + \bar{\tau} \tilde{\Pi}_{\bar{c}} \quad (2.24)$$

and $F_{\tau} = d\bar{c} - \tau d\bar{b}$. The duality group $SL(2, \mathbb{Z})$ acts as follows

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \quad \tau' = \frac{a\tau + b}{c\tau + d} \quad \text{and} \quad \begin{pmatrix} \tilde{\Pi}_{\bar{c}} \\ \tilde{\Pi}_{\bar{b}} \end{pmatrix}' = \Lambda^{-1} \begin{pmatrix} \tilde{\Pi}_{\bar{c}} \\ \tilde{\Pi}_{\bar{b}} \end{pmatrix}, \quad \begin{pmatrix} \bar{F} \\ \bar{H} \end{pmatrix}' = \Lambda \begin{pmatrix} \bar{F} \\ \bar{H} \end{pmatrix}; \quad (2.25)$$

$$\tilde{\Pi}'_{\tau'} = (c\tau + d)^{-1} \tilde{\Pi}_{\tau} \quad \text{and} \quad F'_{\tau'} = (c\tau + d)^{-1} F_{\tau}.$$

3 Gauge group. Classical and Quantum Gauss Laws

The original 5-dimensional action (2.12a) is invariant under the following gauge transformations

$$C_2 \mapsto C_2 + \omega_C, \quad B_2 \mapsto B_2 + \omega_B$$

where ω_C and ω_B are closed 2-forms with integral periods on X_5 . Reduction of these gauge transformations to $\mathbb{R} \times M_4$ yields

$$\begin{aligned} \bar{c} &\mapsto \bar{c} + \omega_c, & \bar{c}_0 &\mapsto \bar{c}_0 + \lambda_c & \text{where } d\omega_c = 0, & \partial_0 \omega_c - d\lambda_c = 0; \\ \bar{b} &\mapsto \bar{b} + \omega_b, & \bar{b}_0 &\mapsto \bar{b}_0 + \lambda_b & \text{where } d\omega_b = 0, & \partial_0 \omega_b - d\lambda_b = 0. \end{aligned} \quad (3.1)$$

If $\omega_c = d\alpha_c$ then the gauge transformation is called small, while if ω_c represents some nontrivial cohomology class in $H^2(M_4, \mathbb{Z})$ the gauge transformation is called large.³

It is easy to see that the momenta $\tilde{\Pi}_{\bar{c}}$ and $\tilde{\Pi}_{\bar{b}}$ are gauge invariant. The parts \mathcal{H}_e and \mathcal{H}_m of the Hamiltonian are separately invariant under the gauge transformations.

3.1 Classical Gauss law. Naive quantization

The part of the Lagrangian (2.18) containing the Lagrangian multipliers \bar{b}_0 and \bar{c}_0 is

$$\mathcal{L}_0 = - \int_{M_4} [\Pi_{\bar{c}} \wedge d\bar{c}_0 + \Pi_{\bar{b}} \wedge d\bar{b}_0] + \pi N \int_{M_4} (\bar{b}_0 \wedge d\bar{c} - \bar{c}_0 \wedge d\bar{b}).$$

The variation with respect b_0 and c_0 yields the classical Gauss law

$$\mathcal{G}_c \equiv \frac{\delta \mathcal{L}_0}{\delta \bar{c}_0} = -d(\Pi_{\bar{c}} + \pi N \bar{b}) = 0 \quad \text{and} \quad \mathcal{G}_b \equiv \frac{\delta \mathcal{L}_0}{\delta \bar{b}_0} = d(-\Pi_{\bar{b}} + \pi N \bar{c}) = 0. \quad (3.2)$$

In the quantum theory the small gauge transformations are generated by this Gauss law. If (α_c, α_b) is a pair of 1-forms then the Gauss law for the small transformations becomes the constraint

$$\Psi(\bar{c}, \bar{b}) = e^{-i \int_{M_4} \alpha_c \wedge \mathcal{G}_c + \alpha_b \wedge \mathcal{G}_b} \Psi(\bar{c}, \bar{b}) = e^{i\pi N \int_{M_4} d\alpha_c \wedge \bar{b} - d\alpha_b \wedge \bar{c}} \Psi(\bar{c} + d\alpha_c, \bar{b} + d\alpha_b). \quad (3.3)$$

on gauge invariant wavefunctions.

How shall we generalize this to the large gauge transformations? The natural guess is to replace $d\alpha$ by ω , a closed 2-form with integral periods. We define

$$U(\omega_c, \omega_b) = \exp \left[2\pi i \int_{M_4} \omega_c \wedge P_c + \omega_b \wedge P_b \right]$$

³A more careful formulation of the B, C fields shows that the underlying gauge invariance is more subtle, but we will not need this level of depth for the present paper. See appendix A for an indication of what is involved.

where

$$P_c = \frac{1}{2\pi} \Pi_{\bar{c}} + \frac{N}{2} \bar{b} \quad \text{and} \quad P_b = \frac{1}{2\pi} \Pi_{\bar{b}} - \frac{N}{2} \bar{c}, \quad (3.4)$$

are conserved “Page charges.” Simple calculation shows that these charges do not commute

$$\left[\int_{M_4} \omega_c \wedge P_c, \int_{M_4} \omega_b \wedge P_b \right] = \frac{iN}{2\pi} \int_{M_4} \omega_c \wedge \omega_b. \quad (3.5)$$

The naive calculation, performed in the same way as the previous one, suggests

$$(U(\omega_c, \omega_b) \Psi)(\bar{c}, \bar{b}) = \exp \left[+i\pi N \int_{M_4} \omega_c \wedge \bar{b} - \omega_b \wedge \bar{c} \right] \Psi(\bar{c} + \omega_c, \bar{b} + \omega_b).$$

Applying U twice one obtains the following group law

$$U(1)U(2) = \exp \left\{ i\pi N \int_{M_4} \omega_c^{(2)} \wedge \omega_b^{(1)} - \omega_b^{(2)} \wedge \omega_c^{(1)} \right\} U(1+2). \quad (3.6)$$

This indicates that there is a \mathbb{Z}_2 anomaly in the quantum Gauss law, suggesting the theory is not consistent for N odd. In fact, the theory *is* consistent for all N , and the above procedure is simply too naive, as it ignores the key geometrical fact that the wavefunction must be considered as a section of a line bundle with nonzero curvature. We will explain this in the next subsection.

The operators P_c, P_b are not gauge invariant. Following the discussion in [4] we define gauge invariant quantities

$$W_c(\phi) = e^{2\pi i \int_{M_4} \phi \wedge P_c} \quad \text{and} \quad W_b(\phi) = e^{2\pi i \int_{M_4} \phi \wedge P_b} \quad (3.7a)$$

where ϕ is an arbitrary 2-form. These operators satisfy the following commutation relation:

$$W_c(\phi_1)W_b(\phi_2) = e^{-2\pi i N \int_{M_4} \phi_1 \wedge \phi_2} W_b(\phi_2)W_c(\phi_1). \quad (3.7b)$$

Under gauge transformations these operators change as follows:

$$\begin{aligned} U(\omega_c, \omega_b)W_c(\phi)U^{-1}(\omega_c, \omega_b) &= W_c(\phi) e^{+2\pi i N \int_{M_4} \phi \wedge \omega_b}; \\ U(\omega_c, \omega_b)W_b(\phi)U^{-1}(\omega_c, \omega_b) &= W_b(\phi) e^{-2\pi i N \int_{M_4} \phi \wedge \omega_c}; \end{aligned} \quad (3.8)$$

One sees that $W_c(\phi)$ and $W_b(\phi)$ are gauge invariant if $N\phi$ is in $\Omega_{\mathbb{Z}}^2(M_4)$. Notice that if ϕ is a closed 2-form with integral periods then the corresponding W ’s are just gauge transformations. Therefore we can identify ϕ with $\phi + \xi$ where ξ is any vector in $H^2(M_4, \mathbb{Z})$. With this identification W_b and W_c generate the finite Heisenberg group W :

$$0 \rightarrow \mathbb{Z}_N \rightarrow W \rightarrow H^2(M_4, \mathbb{Z}_N) \times H^2(M_4, \mathbb{Z}_N) \rightarrow 0.$$

The Hilbert space of the theory should be a representation of this group, and we will confirm this below. W is the magnetic translation group analogous to that of the M -theory 3-form in [4]. From the dual gauge theory point of view W is related to the ’t Hooft lattice of the discrete electric and magnetic charges.

3.2 Quantum Gauss Law

In equation (3.6) we found a potential anomaly in the Gauss law. The resolution of this problem lies in the fact that the wavefunction must be considered as a section of a line bundle \mathcal{L}_N over the space of gauge-invariant field configurations satisfying the Gauss law.⁴ In this section we assume $H^*(M_4, \mathbb{Z})$ is torsion free. The generalization to the case with torsion is indicated in appendix A. Thus, in this section, the Gauss law shows that B_2, C_2 are globally well-defined

The wave function is a section of a line bundle \mathcal{L}_N over the space of pairs of 2-forms $\Omega^2(M_4) \times \Omega^2(M_4)$. This line bundle has a natural connection defined by the phase (2.13). Consider path $p(t) = (C(t), B(t))$ in the space of forms, $t \in [0, 1]$ is the coordinate on the path. Then the parallel transport is defined by (2.13):

$$U(p) = \Phi_B(C(t), B(t)) \in \text{Hom}(\mathcal{L}_N|_{(C(0), B(0))}, \mathcal{L}_N|_{(C(1), B(1))}) \quad (3.9)$$

It is straightforward to compute the curvature of (3.9)

$$\Omega((\phi_c^{(1)}, \phi_b^{(1)}), (\phi_c^{(2)}, \phi_b^{(2)})) = 2\pi i N \int_{M_4} (\phi_c^{(1)} \wedge \phi_b^{(2)} - \phi_b^{(1)} \wedge \phi_c^{(2)}) \quad (3.10)$$

where $\phi_b^{(i)}, \phi_c^{(i)}$ are arbitrary 2-forms. Now, for any 2-forms ϕ_b, ϕ_c introduce the straightline path

$$p_{\bar{c}, \bar{b}; \phi_c, \phi_b}(t) = \{C(t) = \bar{c} + t\phi_c, B(t) = \bar{b} + t\phi_b\} \quad (3.11)$$

Using the formula for the curvature we find

$$U(p_{\bar{c}, \bar{b}; \phi^{(1)} + \phi^{(2)}}) = U(p_{\bar{c} + \phi_c^{(1)}, \bar{b} + \phi_b^{(1)}; \phi^{(2)}}) U(p_{\bar{c}, \bar{b}; \phi^{(1)}}) \exp\left\{i\pi N \int_{M_4} (\phi_c^{(1)} \wedge \phi_b^{(2)} - \phi_b^{(1)} \wedge \phi_c^{(2)})\right\}. \quad (3.12)$$

It follows from (3.12) that parallel transport *does not* define a lift of the gauge group to the total space of \mathcal{L}_N . To define the lift of the group action we choose the standard path, say (3.11). Then define the action on a section Ψ of \mathcal{L}_N by

$$(g(\omega_c, \omega_b) \cdot \Psi)(\bar{c} + \omega_c, \bar{b} + \omega_b) = \varphi(\bar{c}, \bar{b}; \omega_c, \omega_b)^* U(p_{\bar{c}, \bar{b}; \omega_c, \omega_b}) \Psi(\bar{c}, \bar{b}) \quad (3.13)$$

where φ is a phase, and ω_c, ω_b are closed 2-forms with integral periods. The “lifting phase” φ must satisfy

$$\varphi(\bar{c}, \bar{b}; \omega^{(1)} + \omega^{(2)}) = \varphi(\bar{c} + \omega_c^{(1)}, \bar{b} + \omega_b^{(1)}; \omega^{(2)}) \varphi(\bar{c}, \bar{b}; \omega^{(1)}) e^{i\pi N \int_{M_4} \omega_c^{(1)} \wedge \omega_b^{(2)} - \omega_b^{(1)} \wedge \omega_c^{(2)}}. \quad (3.14)$$

⁴The following discussion is closely related to section 6 in [15].

Since we are working in the case where $H^*(M_4, \mathbb{Z})$ is torsion free the lifting phase can be written in terms of local integrals. The most general solution of (3.14) satisfying $\varphi(\bar{c}, \bar{b}; 0, 0) = 1$ is:

$$\varphi(\bar{c}, \bar{b}; \omega_c, \omega_b) = \exp \left\{ i\pi\rho \int_{M_4} \omega_c \wedge \omega_b + i\pi\alpha \int_{M_4} \bar{b} \wedge \omega_c - i\pi\beta \int_{M_4} \bar{c} \wedge \omega_b \right\}$$

where $\beta \equiv N - \rho \pmod{2\mathbb{Z}}$ and $\alpha \equiv N + \rho \pmod{2\mathbb{Z}}$.

We now trivialize the bundle \mathcal{L}_N by using parallel transport along the paths (3.11) to define a canonical nowhere vanishing section $S(\bar{c}, \bar{b})$. The *ratio* $\psi(\bar{c}, \bar{b}) := \Psi(\bar{c}, \bar{b})/S(\bar{c}, \bar{b})$ is a function, rather than a section. The action of the gauge group on this function is

$$(\mathfrak{g}(-\omega_c, -\omega_b) \cdot \psi)(\bar{c}, \bar{b}) = \varphi^*(-\omega_c, -\omega_b; c + \omega_c, b + \omega_b) \exp \left[-i\pi N \int_{M_4} \bar{b} \wedge \omega_c - \bar{c} \wedge \omega_b \right] \psi(\bar{c} + \omega_c, \bar{b} + \omega_b).$$

This action of the gauge group must agree with (3.3) for $\omega_c = d\alpha_c$, $\omega_b = d\alpha_b$, therefore one concludes that $\alpha = \beta = 2N$, and

$$\varphi(\bar{c}, \bar{b}; \omega_c, \omega_b) = e^{-i\pi N \int_{M_4} \omega_c \wedge \omega_b + 2\pi i N \int_{M_4} \bar{b} \wedge \omega_c - \bar{c} \wedge \omega_b}. \quad (3.15)$$

Thus the gauge transformations are given by

$$(\mathfrak{g}(-\omega_c, -\omega_b) \cdot \psi)(c, b) = e_{\omega_c, \omega_b}^*(c, b) \psi(c + \omega_c, b + \omega_b) \quad (3.16a)$$

where

$$e_{\omega_c, \omega_b}(c, b) = \exp \left[-i\pi N \int_{M_4} \omega_b \wedge \omega_c - i\pi N \int_{M_4} \bar{b} \wedge \omega_c - \bar{c} \wedge \omega_b \right] \quad (3.16b)$$

The Gauss law $g \cdot \Psi(c, b) = \Psi(g \cdot (c, b))$ takes the following form

$$\psi^{\text{phys}}(\bar{c} + \omega_c, \bar{b} + \omega_b) = e_{\omega_c, \omega_b}(c, b) \psi^{\text{phys}}(\bar{c}, \bar{b}). \quad (3.17)$$

Under an $SL(2, \mathbb{Z})$ transformation by Λ (2.25) the cocycle (3.15) transforms as

$$\varphi(\Lambda \cdot (c, b); \Lambda \cdot (\omega_c, \omega_b)) = \varphi(c, b; \omega_c, \omega_b) e^{i\pi N \int_{M_4} ac\omega_c \wedge \omega_c + bd\omega_b \wedge \omega_b}. \quad (3.18)$$

This appears to break the $SL(2, \mathbb{Z})$ invariance. However if M_4 is a spin manifold, then the index theorem tells us that $\int_{M_4} \omega \wedge \omega \in 2\mathbb{Z}$ for $\omega \in H^2(M_4, \mathbb{Z})$, and therefore the exponential factor is 1. So we require M_4 to be spin manifold. Put differently, if M_4 is not spin then $SL(2, \mathbb{Z})$ does not commute with the gauge projection. This is in accord with [1]. In the case that M_4 is not spin we expect that the theory can be modified to restore $SL(2, \mathbb{Z})$ invariance, for reasons described below, but we leave this for the future.

4 Spectrum in the harmonic sector

Using the Hodge decomposition we can rewrite \bar{b} , \bar{c} as

$$\bar{c} = c^h + c' + c'' \quad \text{and} \quad \bar{b} = b^h + b' + b'' \quad (4.1)$$

where $c^h, b^h \in \text{Harm}^2(M_4, \mathbb{R})$, c', b' are projections on image of d^\dagger , and c'', b'' are projections on image of d . The operators dd^\dagger and $d^\dagger d$ are separately self adjoint with respect to the metric $\langle \alpha_p, \beta_p \rangle = \int_{M_4} \alpha_p * \beta_p$. Moreover they are orthogonal with respect to this inner product. Therefore the space of 2-forms decomposes as $\Omega^2(M_4) = \text{Harm}^2(M_4) \oplus \text{im}(dd^\dagger, \Omega^2) \oplus \text{im}(d^\dagger d, \Omega^2)$.

As we will see in a moment there is a factorization of the Hamiltonian on the harmonic Hamiltonian and the one corresponding to the massive modes. Therefore the wave function Ψ also factorizes as $\Psi = \Psi_{\text{harm}} \Psi_{\text{massive}}$. The harmonic Hamiltonian does not depend on time, and hence we can assume that Ψ_{harm} is its eigenfunction. The massive sector has a unique groundstate, but the harmonic sector has many groundstates, leading to a Hilbert space of “conformal blocks.” We will mostly be focussing on the harmonic sector in what follows.

4.1 Basis

We choose a basis ω^α for $\text{Harm}_{\mathbb{Z}}^2(M_4)$ and ω^n for the orthogonal complement. $\omega^{n'}$ forms a basis for $\text{im}(d^\dagger d)$, and $\omega^{n''}$ forms a basis for $\text{im}(dd^\dagger)$. We can choose $\omega^{n'}$ to be eigenvectors of $d^\dagger d$, then $*\omega^{n'}$ are eigenvectors of dd^\dagger . We define a dual basis by

$$\int_{M_4} \omega^\alpha \wedge \hat{\omega}_\beta = \delta^\alpha_\beta \quad \text{and} \quad \int_{M_4} \omega^n \wedge \hat{\omega}_m = \delta^n_m. \quad (4.2)$$

So we can expand the fields in this basis

$$\begin{aligned} \bar{c} &= c_\alpha \omega^\alpha + c_n \omega^n, & \bar{b} &= b_\alpha \omega^\alpha + b_n \omega^n; \\ \Pi_{\bar{c}} &= \Pi_c^\alpha \hat{\omega}_\alpha + \Pi_c^n \hat{\omega}_n, & \Pi_{\bar{b}} &= \Pi_b^\alpha \hat{\omega}_\alpha + \Pi_b^n \hat{\omega}_n \end{aligned} \quad (4.3)$$

We also have metrics

$$h_{\alpha\beta} = \int_{M_4} \hat{\omega}_\alpha \wedge * \hat{\omega}_\beta \quad \text{and} \quad h^{\alpha\beta} = \int_{M_4} \omega^\alpha \wedge * \omega^\beta \quad (4.4)$$

which are inverse of each other, and the period matrix $\tau^{\alpha\beta} = \int_{M_4} \omega^\alpha \wedge \omega^\beta$. If $\{\omega^\alpha\}$ is an integral basis in $H^2(M_4, \mathbb{Z})$ then the matrix $\tau^{\alpha\beta}$ has two main properties: $\det \tau = 1$, and both the intersection matrix $\tau^{\alpha\beta}$ and its inverse $\tau_{\alpha\beta}$ have integer coefficients.

One sees that the dual basis is $\hat{\omega}_\alpha = (\tau^{-1})_{\alpha\beta} \omega^\beta$, so

$$\Pi_c^\alpha = \tilde{\Pi}_c^\alpha + \pi N b_\beta \tau^{\beta\alpha} \quad \text{and} \quad \Pi_b^\alpha = \tilde{\Pi}_b^\alpha - \pi N c_\beta \tau^{\beta\alpha}.$$

The symplectic form (2.21) becomes $\Omega = \delta\Pi_{\bar{c}}^\alpha \wedge \delta c_\alpha + \delta\Pi_{\bar{c}}^n \wedge \delta c_n + \delta\Pi_b^\alpha \wedge \delta b_\alpha + \delta\Pi_b^n \wedge \delta b_n$ and hence

$$[\Pi_{\bar{c}}^\alpha(t), c_\beta(t)] = -i\delta_\beta^\alpha \quad \text{and} \quad [\Pi_b^\alpha(t), b_\beta(t)] = -i\delta_\beta^\alpha, \quad (4.5)$$

and similarly for $\Pi^{n'}$ and $\Pi^{n''}$.

We can choose a basis of harmonic forms on M_4 in which Hodge $*$ -operator acts diagonally:

$$*\omega^\alpha = +\omega^\alpha, \quad \alpha = 1, \dots, b_2^+; \quad (4.6a)$$

$$*\omega^\alpha = -\omega^\alpha, \quad \alpha = b_2^+ + 1, \dots, b_2^+ + b_2^- = b_2(M_4). \quad (4.6b)$$

Notice that in the basis (4.6) the period matrix is not integral, but is related to the Hodge metric

$$h^{\alpha\beta} = \begin{pmatrix} \tau_+^{\alpha\beta} & 0 \\ 0 & -\tau_-^{\alpha\beta} \end{pmatrix}, \quad \tau^{\alpha\beta} = \begin{pmatrix} \tau_+^{\alpha\beta} & 0 \\ 0 & \tau_-^{\alpha\beta} \end{pmatrix}, \quad \tau_+^{\alpha\beta} > 0 \quad \text{and} \quad \tau_-^{\alpha\beta} < 0. \quad (4.7)$$

The Hamiltonian takes the form

$$\mathcal{H}_e = \frac{1}{4\nu\tau_2} \left[h_{\alpha\beta} (\tilde{\Pi}_\tau^\alpha \tilde{\Pi}_{\bar{\tau}}^\beta + \tilde{\Pi}_{\bar{\tau}}^\alpha \tilde{\Pi}_\tau^\beta) + h_{nm} (\tilde{\Pi}_\tau^n \tilde{\Pi}_{\bar{\tau}}^m + \tilde{\Pi}_{\bar{\tau}}^n \tilde{\Pi}_\tau^m) \right]; \quad (4.8a)$$

$$\mathcal{H}_m = \frac{\nu}{2\Omega^2(t)\tau_2} h^{n'm'} \lambda_{m'} F_{\tau, n'} F_{\bar{\tau}, m'} \quad (4.8b)$$

where $\lambda_{m'}$ are eigenvalues of $d^\dagger d$: $d^\dagger d\omega_m = \lambda_m \omega_m$. One sees that the wave function factorizes on the product of the wave function ψ_h depending on the harmonic modes (c_α, b_α) and the wave function ψ_m depending on the massive modes $(c_{n'}, b_{n'})$. The harmonic Hamiltonian $\mathcal{H}_{\text{harm}}$ is defined by the first term in (4.8a).

Using the commutation relations (4.5) one obtains

$$[\tilde{\Pi}_\tau^\alpha, \tilde{\Pi}_\tau^\beta] = 0, \quad [\tilde{\Pi}_{\bar{\tau}}^\alpha, \tilde{\Pi}_{\bar{\tau}}^\beta] = 0, \quad [\tilde{\Pi}_\tau^\alpha, \tilde{\Pi}_{\bar{\tau}}^\beta] = 4\pi N \tau_2 \tau^{\alpha\beta}. \quad (4.9)$$

In the basis (4.6) the matrix $\tau^{\alpha\beta}$ has block diagonal form (4.7). Assuming that N is positive one sees that for $\alpha = 1, \dots, b_2^+$ the operators $\tilde{\Pi}_\tau^\alpha$ are annihilation operators, while for $\alpha = b_2^+ + 1, \dots, b_2$ the operators $\tilde{\Pi}_{\bar{\tau}}^\alpha$ are annihilation operators. The first term in (4.8a) takes the form

$$\mathcal{H}_{\text{harm}} \stackrel{4.9}{=} \frac{1}{2\nu\tau_2} \left\{ \sum_{\alpha, \beta=1}^{b_2^+} (\tau_+^{-1})_{\alpha\beta} \tilde{\Pi}_{\bar{\tau}}^\alpha \tilde{\Pi}_\tau^\beta - \sum_{\alpha, \beta=b_2^++1}^{b_2} (\tau_-^{-1})_{\alpha\beta} \tilde{\Pi}_\tau^\alpha \tilde{\Pi}_{\bar{\tau}}^\beta \right\} + \frac{\pi N}{\nu} b_2(M_4) \quad (4.10)$$

From this and the block-diagonal form of the metric $h_{\alpha\beta}$ it is easy to see that the ground state function $\Psi_0(b_\alpha, c_\alpha)$ must satisfy

$$\tilde{\Pi}_\tau^\alpha \Psi_0 = 0, \quad \alpha = 1, \dots, b_2^+; \quad \tilde{\Pi}_{\bar{\tau}}^\alpha \Psi_0 = 0, \quad \alpha = b_2^+ + 1, \dots, b_2. \quad (4.11)$$

The ground state energy is $\frac{\pi N}{\nu} b_2(M_4)$ and depends only on the topology of M_4 . The excited states are constructed by acting by the creation operators. The commutation relations of the momenta and Hamiltonian are

$$[\mathcal{H}_{\text{harm}}, \tilde{\Pi}_{\tau}^{\alpha}] = -\frac{2\pi N}{\nu} s_{\alpha} \tilde{\Pi}_{\tau}^{\alpha} \quad \text{and} \quad [\mathcal{H}_{\text{harm}}, \tilde{\Pi}_{\bar{\tau}}^{\alpha}] = \frac{2\pi N}{\nu} s_{\alpha} \tilde{\Pi}_{\bar{\tau}}^{\alpha} \quad (4.12)$$

where $s_{\alpha} = (1_{b_2^+}, -1_{b_2^-})$. Therefore the spectrum of this Hamiltonian is equally gapped with the gap width $2\pi N/\nu$.

The most general solution of equations (4.11) is

$$\psi_0(c^h, b^h) = \exp\left[-\frac{\pi N}{2\tau_2} \int_{M_4} (c^h - \tau b^h) *_4 (c^h - \bar{\tau} b^h)\right] \phi(c_+ - \tau b_+, c_- - \bar{\tau} b_-). \quad (4.13)$$

where $*c_{\pm} = \pm c_{\pm}$ and ϕ is holomorphic.

We now introduce an overcomplete basis by choosing ϕ to be a linear exponential ϕ_{v_c, v_b} . Covariance with respect to $SL(2, \mathbb{Z})$ suggests the following choice

$$\phi_{v_c, v_b}(c_+ - \tau b_+, c_- - \bar{\tau} b_-) = e^{-\frac{\pi N}{\tau_2} \int_{M_4} (v_c - \bar{\tau} v_b)_+ \wedge (c - \tau b)_+ + \frac{\pi N}{\tau_2} \int_{M_4} (v_c - \tau v_b)_- \wedge (c - \bar{\tau} b)_-}. \quad (4.14)$$

4.2 Averaging over the gauge group

To obtain the wave function satisfying the Gauss law (3.17) it is sufficient to average the solution (4.13) over the large gauge transformations (3.16). Hence using ϕ_{v_c, v_b} of (4.14) in (4.13) the physical wave function is

$$\psi_{v_c, v_b}^{\text{phys}}(c^h, b^h) := \sum_{\omega_c, \omega_b \in \Lambda} e_{\omega_c, \omega_b}^*(c^h, b^h) \psi_{v_c, v_b}(c^h + \omega_c, b^h + \omega_b) \quad (4.15)$$

where $\Lambda = \text{Harm}_{\mathbb{Z}}^2(M_4)$ is the lattice of the harmonic 2-forms with integral periods. We now follow a standard procedure and use Poisson resummation to split this sum in a form so that we can extract the conformal blocks. The details are in appendix B. One finds that the sum (4.15), up to an overall normalization independent of b and c , can be written as

$$\Psi_{v_c, v_b}^{\text{phys}}(c, b) = \sum_{\beta \in \Lambda/\Lambda_N} \Psi_{\beta}^{\text{phys}}(c, b; \tau) \Psi_{-\beta}^{\text{phys}}(-v_c, v_b; -\bar{\tau}).$$

where $\Lambda_N \approx \text{Harm}_{N\mathbb{Z}}^2(M_4)$. The physical wave function is thereby found to be

$$\Psi_{\beta}^{\text{phys}}(c, b; \tau) = \mathcal{N}(\tau) \Theta_{\Lambda + \frac{1}{N}\beta, N/2}(\tau, c, b; *) \quad (4.16)$$

where c and b are harmonic 2-forms, and $\mathcal{N}(\tau)$ is a normalization constant which will be fixed later. The Siegel-Narain Θ -function at level k with characteristics c, b is given by the following series

$$\Theta_{\Lambda+\gamma, k}(\tau, c, b; *) = e^{2\pi i k \int_{M_4} c \wedge b} \sum_{\omega \in \Lambda + \gamma} e^{2\pi i k \tau \int_{M_4} (\omega+b)_+^2 + 2\pi i k \bar{\tau} \int_{M_4} (\omega+b)_-^2 - 4\pi i k \int_{M_4} c \wedge (\omega+b)} \quad (4.17)$$

where γ is an element of $\frac{1}{2k}\Lambda$.

The magnetic translation group (3.7) acts on the physical wave functions as follows:

$$W_c(\phi) \Psi_\beta^{\text{phys}}(c, b; \tau) = e^{-2\pi i \int_{M_4} \phi \wedge \beta} \Psi_\beta^{\text{phys}}(c, b; \tau); \quad (4.18a)$$

$$W_b(\phi) \Psi_\beta^{\text{phys}}(c, b; \tau) = \Psi_{\beta+N\phi}^{\text{phys}}(c, b; \tau). \quad (4.18b)$$

Here it is assumed that $N\phi \in \Lambda$. One sees that the space of the physical wave functions is a representation space for W . In [1] the algebra (3.7) and its representation (4.18) occurs. The description of the operators in [1], (eq. 3.5) compared to our (3.7) is different. This happens because we retain the kinetic terms for the b, c fields. It is easy to see that in the limit $\nu \rightarrow 0$ the operators W_c and W_b becomes the operators defined in [1]. In making this comparison one must regard the cohomology class $[N\phi]$ as Poincaré dual to a 2-cycle in M_4 .

4.3 Normalization of the wave function

Since the Hamiltonian and Hilbert space factorize into flat and massive sectors we can consider the wavefunction restricted to the flat fields. We now observe an interesting consequence of normalizing the wavefunction of the flat fields.

The inner product on the space of flat fields is defined by

$$\langle \Psi_\beta, \Psi_{\beta'} \rangle := \int_{Z^2(M_4, \mathbb{R}) \times Z^2(M_4, \mathbb{R})} \frac{\mathcal{D}_g C \mathcal{D}_g B}{\text{vol}(\text{gauge group})} \overline{\Psi_\beta(B, C)} \Psi_{\beta'}(B, C), \quad (4.19)$$

where the integral runs over all closed 2-forms on M_4 . The integral descends to one on the space of gauge inequivalent flat fields. This space is $\mathbb{T} \times \mathbb{T}$ where $\mathbb{T} = H_{DR}^2(M_4, \mathbb{R}) / H_{DR}^2(M_4, \mathbb{Z})$. In order to fix the gauge we will follow the recipe of [18].

We use the Hodge decomposition to write $C = c^h + d\alpha_c$ and $B = b^h + d\alpha_b$, where c^h and b^h are harmonic 2-forms, α_b and α_c are 1-forms. However α_b and α_c also have gauge degrees of freedom. We can fix this gauge freedom by saying that $\alpha_b = \alpha_b^T$ and $\alpha_c = \alpha_c^T$ are in the image of d^\dagger . The measure as usual can be obtained from the norm:

$$\|\delta C\|_g^2 = \int_{M_4} \delta C * \delta C = \int_{M_4} \delta c^h * \delta c^h + \int_{M_4} \delta \alpha_c^T * (d^\dagger d) \delta \alpha_c^T \quad \Rightarrow \quad \mathcal{D}_g C = \sqrt{\det'_1(d^\dagger d)} \mathcal{D}_g c^h \mathcal{D}_g \alpha_c^T,$$

where $\det'_1(d^\dagger d)$ is the determinant of the operator $d^\dagger d$ on the space of 1-forms, and $'$ means that we excluded the zero modes. Using some identities [18] one can rewrite it as the ratio of the Laplacian operators $\det' \Delta_1 / \det' \Delta_0$. The integrals over $\mathcal{D}_g \alpha_c^T$ and $\mathcal{D}_g \alpha_b^T$ partially cancel the volume of the small gauge transformations, the mismatch coming from the ghosts for ghosts phenomenon is the factor $(\det' \Delta_0)^{-1}$. The volume of the large gauge transformations is cancelled by restricting the integral over $\text{Harm}^2(M_4, \mathbb{R})$ to integral over the Jacobian $\mathbb{T} = \text{Harm}^2(M_4, \mathbb{R}) / \text{Harm}_{\mathbb{Z}}^2(M_4)$. So after the the gauge fixing one obtains the following expression for the norm:

$$\langle \Psi_\beta, \Psi_{\beta'} \rangle = \frac{\det' \Delta_1}{(\det' \Delta_0)^2} \int_{\mathbb{T} \times \mathbb{T}} \mathcal{D}_g c^h \mathcal{D}_g b^h \overline{\Psi_\beta(b^h, c^h)} \Psi_{\beta'}(b^h, c^h). \quad (4.20)$$

Now we substitute (4.16) into (4.20). The field c^h appears linearly in the exponential, the integral over c^h yields two Kronecker symbols $\delta_{\beta\beta'}$ and $\delta_{\omega, \omega'}$ where ω and ω' are summation variables in the definition of the Θ -function:

$$\langle \Psi_\beta^{\text{phys}}, \Psi_{\beta'}^{\text{phys}} \rangle = \delta_{\beta, \beta'} |\mathcal{N}(\tau)|^2 \frac{\det' \Delta_1}{(\det' \Delta_0)^2} \sum_{\omega \in \Lambda} \int_{\mathbb{T}} \mathcal{D}_g b^h e^{-2\pi N \tau_2 \int_{M_4} (b^h + \omega + \frac{1}{N} \beta) * (b^h + \omega + \frac{1}{N} \beta)}.$$

The sum over ω combines with the integral over \mathbb{T} to the integral over $\text{Harm}^2(M_4, \mathbb{R})$. This Gaussian integral is easy to calculate, and one finally finds that the normalization constant $\mathcal{N}(\tau)$ is

$$\mathcal{N}(\tau) = (2N\tau_2)^{b_2/4} \det' \Delta_0 (\det' \Delta_1)^{-1/2}. \quad (4.21)$$

Notice that $\mathcal{N}(-1/\tau) = |\tau|^{-b_2/2} \mathcal{N}(\tau)$.

4.4 Representation of the duality group

The $SL(2, \mathbb{Z})$ group is realized as follows:

$$T : \quad \Psi_\beta^{\text{phys}}(c + b + \frac{1}{2}w_2, b; \tau + 1) = e^{\frac{i\pi}{N}(\beta, \beta) - i\pi(w_2, \beta) - \frac{i\pi N}{2}(w_2, b)} \Psi_\beta^{\text{phys}}(c, b; \tau); \quad (4.22a)$$

$$S : \quad \Psi_\beta^{\text{phys}}(b, -c; -1/\tau) = (-i\tau)^{\sigma/4} (i\bar{\tau})^{-\sigma/4} N^{-b_2/2} \sum_{\beta' \in \Lambda/\Lambda_N} e^{-\frac{2\pi i}{N}(\beta, \beta')} \Psi_{\beta'}^{\text{phys}}(c, b; \tau). \quad (4.22b)$$

Here $\sigma = b_2^+ - b_2^-$ is the Hirzebruch signature, N^{b_2} is the order of the finite group Λ/Λ_N . One sees that the physical wave functions are modular forms of weight $(\sigma, -\sigma)$. Here w_2 is a characteristic vector, such that

$$(\omega, \omega) = (\omega, w_2) \pmod{2}.$$

For a spin manifold it equals zero.

4.5 Dual conformal field theory

Finally, one can interpret the Θ -function in (4.16) as a path integral over the $U(1)$ gauge field A with $F = dA$. We normalize A such that F has integral periods. The dual action in the topological sector can be written as

$$e^{iI_{\text{dual}}(F, \tau; c, b)} = e^{i\pi N \int_{M_4} c \wedge b - 2\pi i N \int_{M_4} c \wedge (F + b + \frac{1}{N}\beta)} \times \exp \left[i\pi N \tau \int_{M_4} (F + b + \frac{1}{N}\beta)_+^2 + i\pi N \bar{\tau} \int_{M_4} (F + b + \frac{1}{N}\beta)_-^2 \right] \quad (4.23a)$$

Here $\beta \in \text{Harm}_{\mathbb{Z}}^2(M_4)$ is a harmonic representative. One sees that the $U(1)$ field F is obtained from the $U(1)$ part of the $U(N)$ gauge theory. $\tau = \frac{\theta}{\pi} + \frac{i}{g_B}$ is the gauge theory coupling constant

An equivalent form, which makes the $SL(2, \mathbb{Z})$ properties manifest is:

$$e^{iI_{\text{dual}}(F, \tau; c, b)} = e^{i\pi N \tau \int_{M_4} (F + \frac{1}{N}\beta)_+^2 - 2\pi i N \int_{M_4} (c - \tau b)_+ \wedge (F + \frac{1}{N}\beta)_+ - i\pi N \int_{M_4} (c - \tau b)_+ \wedge b_+} \times e^{i\pi N \bar{\tau} \int_{M_4} (F + \frac{1}{N}\beta)_-^2 - 2\pi i N \int_{M_4} (c - \bar{\tau} b)_- \wedge (F + \frac{1}{N}\beta)_- - i\pi N \int_{M_4} (c - \bar{\tau} b)_- \wedge b_-}. \quad (4.23b)$$

Note that F_+ transforms as a modular form of weight 1 and hence must couple to $(c - \tau b)$ which transforms with weight -1 .

It is very interesting that the normalization (4.21) of the bulk-theory wavefunction is precisely the one-loop determinant of the gauge boson. Thus we confirm the AdS/CFT correspondence at the full quantum level for this free field sector. It would be interesting to give a physical interpretation to the wavefunction overlaps (4.19) from the bulk supergravity viewpoint.

As we have stressed, we are assuming M_4 is spin in this paper. If it is not spin then (4.23a) can be modified to make the theory $SL(2, \mathbb{Z})$ invariant. One must shift the quantization law of F by $w_2(M_4)$ and add a phase factor $\exp[i\pi \int F w_2(M_4)]$. Then the path integral will be $SL(2, \mathbb{Z})$ invariant.⁵ This indicates that the B, C theory can also be suitably modified to restore $SL(2, \mathbb{Z})$ invariance in the non-spin case. We suspect that it is related to further subtleties in the IIB phase, but we leave this for the future.

5 't Hooft and Wilson Lines

In this section we analyze how the conformal blocks of the dual $U(1)$ gauge theory changes in the presence of Wilson surfaces for B and C fields [1, 16, 17, 21]. These are usually denoted

$$\exp[2\pi i \int_{\Sigma_B} B_2 - 2\pi i \int_{\Sigma_C} C_2] \quad (5.1)$$

⁵G.M. Thanks J. Evslin and E. Witten for a useful conversation on this point.

where Σ_B and Σ_C are 2-manifolds in X_5 . We will denote this factor by

$$\text{Hol}_B(\Sigma_B; \gamma_B) \text{Hol}_C^*(\Sigma_C; \gamma_C), \quad (5.2)$$

where the boundary of Σ_B, Σ_C is a 1-manifold γ_B, γ_C in M_4 . The surfaces Σ_B, Σ_C need not be connected but for simplicity of notation we will assume below that they, and γ_B, γ_C are connected. For a closed surface Σ_B the holonomy is a complex number, and the following variational formula holds:

$$\text{Hol}_{B+b_2}(\Sigma_B) = \text{Hol}_B(\Sigma_B) e^{2\pi i \int_{\Sigma_B} b_2} \quad (5.3)$$

where b_2 is a globally well defined 2-form. For surfaces with boundary the holonomy is a section of a line bundle over $\Omega^2(M_4) \times Z_1(M_4)$. This line bundle is defined by the usual gluing law: Let (\tilde{B}, Σ_B) and (\tilde{B}', Σ'_B) be two extensions of the contour $\gamma_B \in Z_1(M_4)$ and B -field $B \in \Omega^2(M_4)$ to X_5 then the holonomies are related by

$$\frac{\text{Hol}_{\tilde{B}}(\Sigma_B; \gamma_B)}{\text{Hol}_{\tilde{B}'}(\Sigma'_B; \gamma_B)} = \text{Hol}_{\tilde{B}-\tilde{B}'}(\Sigma_B \bar{\Sigma}'_B). \quad (5.4)$$

The phase on the right hand side is the holonomy of the B -field around the closed surface $\Sigma_B \bar{\Sigma}'_B$ which is obtained by gluing Σ_B and $\bar{\Sigma}'_B$ along the boundary.

Inclusion of the holonomies (5.2) modifies the line bundle in which wave functions lives. The new line bundle is over the space $\Omega^2(M_4) \times \Omega^2(M_4) \times Z_1(M_4) \times Z_1(M_4)$. A connection on this line bundle is defined as follows: we choose a path $p(t) = (C(t), B(t))$ and extensions $\Sigma_C(t) = (\gamma_C(t), t)$, $\Sigma_B(t) = (\gamma_B(t), t)$ of the loops γ_C and γ_B , then define

$$U(p(t), \Sigma_B(t), \Sigma_C(t)) = \Phi(p(t); M_4 \times [0, 1]) \times \text{Hol}_{B(t)}(\Sigma_B; \partial \Sigma_B) \text{Hol}_{C(t)}^*(\Sigma_C; \partial \Sigma_C). \quad (5.5)$$

It is straightforward to compute the curvature of (5.5), the components along $\Omega^2(M_4) \times \Omega^2(M_4)$ are given by (3.10) as before, while the components along $Z_1(M) \times Z_1(M)$ are given by $\int_{\gamma_B} H - \int_{\gamma_C} F$ (where H, F are the fieldstrengths of a family of 2-form connections over $Z_1(M) \times Z_1(M)$).

It is easy to see that for the straightline path (3.11) the composition of the parallel transports is given by Eq. (3.12). We can still choose the cocycle (3.15) to define the group lift. To define the action of the gauge group on the wave functions, we must first trivialize the line bundle. To this end we first choose the reference point (C_\bullet, B_\bullet) . Then any field in this cohomology class can be written as $(C_\bullet + \bar{c}, B_\bullet + \bar{b})$. We also have to choose base contours $(\gamma_C^\bullet, \gamma_B^\bullet)$ which represent some homology class in $H_1(M_4, \mathbb{Z}) \times H_1(M_4, \mathbb{Z})$. Then an arbitrary element (γ_C, γ_B) from the same homology class is related to base curve by adding a 2-chain (D_c, D_b) where $(\partial D_c, \partial D_b) = (\gamma_C - \gamma_C^\bullet, \gamma_B - \gamma_B^\bullet)$. Now we can proceed as in section 3.2, we choose a standard nowhere vanishing section by the parallel

transport. Then the wave function ψ is defined as the ratio of a section Ψ and the standard section. This leads to the following modification of the Gauss law:

$$\psi(\bar{c}, \bar{b}) = e_{\omega_c, \omega_b}^* (\bar{c}, \bar{b}) e^{-2\pi i \int_{D_c} \omega_c + 2\pi i \int_{D_b} \omega_b} \psi(\bar{c} + \omega_c, \bar{b} + \omega_b). \quad (5.6)$$

As a cross check let us compare this Gauss law with the classical one. Near the boundary the surfaces Σ_B and Σ_C look like direct products $\mathbb{R}_+ \times \gamma_B$ and $\mathbb{R}_+ \times \gamma_C$. It is easy to see that the Gauss law (3.2) is modified to:

$$\mathcal{G}_c = -d(\Pi_c + \pi N \bar{b}) + 2\pi \delta(\gamma_C) = 0 \quad \text{and} \quad \mathcal{G}_c = d(-\Pi_b + \pi N \bar{c}) - 2\pi \delta(\gamma_B) = 0. \quad (5.7)$$

Similarly to (3.3) this yields

$$\psi(\bar{c}, \bar{b}) = e^{i\pi N \int_{M_4} d\alpha_c \wedge \bar{b} - d\alpha_b \wedge \bar{c}} e^{-2\pi i \int_{\gamma_C} \alpha_c + 2\pi i \int_{\gamma_B} \alpha_b} \psi(\bar{c} + d\alpha_c, \bar{b} + d\alpha_b)$$

which agrees with (5.6) for $\omega_c = d\alpha_c$ and $\omega_b = d\alpha_b$. Clearly the Hamiltonian near the boundary does not change in the presence of Wilson surfaces. Therefore we can take the solution (4.13) and average it over the modified large gauge transformations (5.6). Using the techniques presented in section 4 and appendix B one finds that the action for the dual gauge theory is

$$e^{iI_{\text{dual}}(F, \tau; c, b; D_c, D_b)} = e^{2\pi i \int_{D_b} (F + \frac{1}{N}\beta)} e^{-i\pi N \int_{M_4} c \wedge b - 2\pi i N \int_{M_4} c \wedge (F - \frac{1}{N}\delta(D_c) + \frac{1}{N}\beta)} \\ \times \exp \left[i\pi N \tau \int_{M_4} (F + b - \frac{1}{N}\delta(D_c) + \frac{1}{N}\beta)_+^2 + i\pi N \bar{\tau} \int_{M_4} (F + b - \frac{1}{N}\delta(D_c) + \frac{1}{N}\beta)_-^2 \right]. \quad (5.8)$$

Here the action includes the Wilson line for A , written as a surface integral over D_b , as expected. The presence of δ -functions indicates the need for some regularization of the self-energy of the 't Hooft operators. Making precise sense of this factor lies beyond the scope of this paper. We can, however, confirm the dependence on the choice of trivialization observed in [1] as follows.

As explained above, a trivialization is related to a choice of D_c and D_b . Let D'_c be a different 2-chain such that $\partial D'_c = \gamma_C - \gamma_C^\bullet$. Consider a 2-cycle $E = D'_c - D_c$. We want to make a change of variable in the path integral of the dual gauge theory (5.8) such that $F \mapsto F - \frac{1}{N}\delta(E)$. We, certainly, can do this if E is a boundary of a 3-manifold Y , then we just shift A by $-\frac{1}{N}\theta(Y)$ where $\theta(Y)$ is the characteristic function of Y . This change of integration variable not only changes D_c to D'_c in (5.8) but also yields an extra factor:

$$\int \mathcal{D}A e^{iI_{\text{dual}}(F, \tau; c, b; D_c, D_b)} = e^{-\frac{2\pi i}{N} \int_{D_b} \delta(E)} \int \mathcal{D}A e^{iI_{\text{dual}}(F, \tau; c, b; D'_c, D_b)}. \quad (5.9)$$

The integral in the first term on the right hand side is an integer which equals the intersection number $\#(D_c \cdot E)$ of the 2-chain D_b with the 2-cycle E . Hence, under a change of trivialization the expectation value of a product of both 't Hooft and Wilson lines is multiplied by an N^{th} root of unity in accord with [1].

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Appendix

A Formulation in terms of Cheeger-Simons characters

In this appendix we indicate briefly how the above paper could be formulated in somewhat more abstract terms using Cheeger-Simons cohomology. Some discussion of this can also be found in [1] section 3.5. The advantage of this formalism is that it also covers the case when the cohomology $H^*(M_4, \mathbb{Z})$ has torsion.

The gauge invariant information in the supergravity potentials B_2, C_2 is properly regarded as a Cheeger-Simons character $\check{B}, \check{C} \in \check{H}^3(M_4)$. For recent reviews of Cheeger-Simons characters in this context see [19, 20, 15]. We follow the notation of [15]. If X_5 is oriented and compact there is a canonical multiplication and integration and the expression

$$\exp[2\pi i N \int_{X_5} \check{B} \cdot \check{C}] \quad (\text{A.1})$$

is a well-defined phase. When X_5 is oriented with boundary M_4 (A.1) must be regarded as a section of a line bundle \mathcal{L}_N over the space

$$\check{H}^3(M_4) \times \check{H}^3(M_4) \quad (\text{A.2})$$

Moreover, \mathcal{L}_N comes together with a canonical connection. (See for example the discussion in section 3.3 above.) There is an Hermitian metric on \mathcal{L}_N so that the connection is unitary. The connection has curvature

$$\Omega = 2\pi N \int_{M_4} \delta B_2 \wedge \delta C_2 \quad (\text{A.3})$$

and is N times the canonical symplectic form on (A.2). However, since we consider the theory with standard kinetic terms the wavefunctions in the quantum theory are in the Hilbert space

$$L^2(\check{H}^3(M_4) \times \check{H}^3(M_4); \mathcal{L}_N) \quad (\text{A.4})$$

and the Hamiltonian is the canonical Laplacian where we use the $SL(2, \mathbb{Z})$ -covariant, translation invariant, metric (B.3) below.

Note that the main difference between [1] and the present discussion is that in [1], (A.2) was considered to be a phase space. In the present discussion it is the configuration space, and the phase space is the cotangent bundle of (A.2).

Now, the space $\check{H}^3(M_4) \times \check{H}^3(M_4)$ is a disjoint union of spaces modelled on $\Omega^2/\Omega_{\mathbb{Z}}^2 \times \Omega^2/\Omega_{\mathbb{Z}}^2$, with the connected components labelled by $H^3(M, \mathbb{Z}) \oplus H^3(M, \mathbb{Z})$. At this point we encounter an interesting subtlety. The space (A.2) actually labels the isomorphism classes of a field with automorphisms, as described in [19, 15, 4]. The gauge fields are objects in a groupoid, and the automorphism group of an object is $H^1(M_4, U(1)) \times H^1(M_4, U(1))$. In order for the line bundle \mathcal{L}_N to be well-defined on (A.2) we require the tadpole condition on the characteristic classes, determined by the requirement that the automorphism group act trivially on \mathcal{L}_N . The action of this automorphism group is given as follows. Suppose $(\check{\chi}_c, \check{\chi}_b)$ are flat characters in $\check{H}^2(M_4) \times \check{H}^2(M_4)$, and let \check{t} be the canonical character on $\check{H}^1(S^1)$. Then the “lifting phase” of section 3.2 is properly defined by

$$\varphi(\check{C}, \check{B}; \check{\chi}_c, \check{\chi}_b) = \Phi_B(\check{B} + \check{t} \cdot \check{\chi}_b, \check{C} + \check{t} \cdot \check{\chi}_c; M_4 \times S^1) \quad (\text{A.5})$$

When restricted to flat characters this is a homomorphism, and if $\check{B}, \check{C} = 0$ then it is Poincaré dual to torsion background charges $\mu_c, \mu_b \in H_T^3(M_4; \mathbb{Z})$, analogous to the class μ discussed in [19] or the class $\Theta(0)$ discussed in [15]. The Gauss law on the characteristic class is

$$a(\check{B}) = \mu_b \quad a(\check{C}) = \mu_c. \quad (\text{A.6})$$

where $a(\check{B})$ denotes the characteristic class of the Cheeger-Simons character.

The condition (A.6) is analogous to the tadpole condition for the M -theory 3-form, given by equation (7.7) of [15], and it arises in the same way. To explain this, let us note parenthetically that it is possible to give an analog of the “ E_8 model for the C -field” for 2-form potentials whose isomorphism class is an element of $\check{H}^3(M)$.⁶ Let G be a compact Lie group whose homotopy type is that of $K(\mathbb{Z}, 3)$ up to the n -skeleton. (For example, $G = E_8$ for $n < 15$.) On a manifold of dimension n we define an object in the groupoid to be a pair (g, b) where $g : M \rightarrow G$ is a smooth map and $b \in \Omega^2(M)$ is a globally defined 2-form. The isomorphism class of (g, b) is the differential character defined by the holonomies

$$\check{\chi}_{g,b}(\Sigma) = e^{2\pi i(WZ(g,\Sigma) + \int_{\Sigma} b)} \quad (\text{A.7})$$

Here $WZ(g, \Sigma)$ is the Wess-Zumino term, thus $WZ(g, \Sigma) = \int_B \text{Tr}(g^{-1}dg)^3$ where $\partial B = \Sigma$ and $\text{Tr}(g^{-1}dg)^3$ is the pullback of a representative of a generator of $H^3(G, \mathbb{Z})$. The field strength is

⁶This formulation suggests a speculation. This model could be applied to the 2-form gauge field on the $M5$ brane, thus reformulating the 5-brane theory as a six-dimensional nonlinear sigma model with target space E_8 . It is then natural to ask if the map to E_8 “becomes dynamical” for coincident 5-branes, in a way analogous to the way the topological E_8 gauge field of M -theory becomes a dynamical gauge field in heterotic M -theory.

$\omega(\check{\chi}_{g,b}) = \text{Tr}(g^{-1}dg)^3 + db$ and the characteristic class is the homotopy class of $g : M \rightarrow G$. We will not give the full description of the morphisms here. It suffices to note that the automorphism group of an object is $H^1(M, U(1))$.

The connection on \mathcal{L}_N is $SL(2, \mathbb{Z})$ invariant and thus $SL(2, \mathbb{Z})$ acts on the Hilbert space, as described in [1]. The translation group of (A.2) on itself is not an invariance of the connection, thus leading to the action of the magnetic translation group, described by the Page charges. This is the Heisenberg group described in equation (3.7) above. The space $\Omega^2/\Omega_{\mathbb{Z}}^2 \times \Omega^2/\Omega_{\mathbb{Z}}^2$ contains the torus $H^2(M, \mathbb{R}/\mathbb{Z}) \times H^2(M, \mathbb{R}/\mathbb{Z})$, and the theta functions in this paper define the appropriate sections of \mathcal{L}_N over this torus. The relevant complex structure and polarization are described in appendix B below.

B Gaussian sums on $\text{Harm}^2(M_4, \mathbb{Z}) \times \text{Harm}^2(M_4, \mathbb{Z})$

B.1 Symplectic structure, complex structure, and metric

Consider the lattice $V_{\mathbb{Z}} = \text{Harm}_{\mathbb{Z}}^2(M_4) \times \text{Harm}_{\mathbb{Z}}^2(M_4)$ of rank $2b_2(M_4)$. This lattice has integral valued symplectic form Ω

$$\Omega((\omega_c, \omega_b), (\omega'_c, \omega'_b)) = \int_{M_4} \begin{pmatrix} \omega_c & \omega_b \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \omega'_c \\ \omega'_b \end{pmatrix}. \quad (\text{B.1})$$

Given a metric on M_4 and a complex number τ with $\tau_2 > 0$ one can define a complex structure J on $V_{\mathbb{R}}$:

$$J \begin{pmatrix} \phi_c \\ \phi_b \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathcal{M}(\tau) \begin{pmatrix} *\phi_c \\ *\phi_b \end{pmatrix} \quad (\text{B.2})$$

where $\mathcal{M}(\tau)$ is defined in (2.12b). It is easy to see that $J^2 = -1$. This complex structure is compatible with the symplectic one

$$\Omega(J \cdot (\phi_c, \phi_b), J \cdot (\phi'_c, \phi'_b)) = \Omega((\phi_c, \phi_b), (\phi'_c, \phi'_b)).$$

In this case we can define a quadratic form

$$g((\phi_c, \phi_b), (\phi'_c, \phi'_b)) = \Omega(J \cdot (\phi_c, \phi_b), (\phi'_c, \phi'_b)) = \int_{M_4} (*\phi_c, *\phi_b) \mathcal{M}(\tau) \begin{pmatrix} \phi'_c \\ \phi'_b \end{pmatrix} \quad (\text{B.3})$$

We can choose a symplectic basis α^I, β_I for $V_{\mathbb{Z}}$ to be

$$\alpha^I = (0, \omega^I) \quad \text{and} \quad \beta_I = (\hat{\omega}_I, 0) \quad (\text{B.4})$$

where ω^I and $\hat{\omega}_I$ are dual bases in $\text{Harm}_{\mathbb{Z}}^2(M_4)$. If $\boldsymbol{\tau}^{IJ}$ is the period matrix in the basis ω^I then $\hat{\omega}_I \boldsymbol{\tau}^{IJ} = \omega^J$. One can verify that basis (B.4) is indeed a symplectic basis

$$\Omega(\alpha^I, \alpha^J) = 0 = \Omega(\beta_I, \beta_J) \quad \text{and} \quad \Omega(\alpha^I, \beta_J) = \delta^I_J.$$

The complex structure (B.2) acts on the basis (B.4) as follows

$$J \cdot \begin{pmatrix} \alpha^I \\ \beta_I \end{pmatrix} = \frac{1}{\tau_2} \begin{pmatrix} -\tau_1(h\boldsymbol{\tau}^{-1})^I_J & -|\tau|^2 h^{IJ} \\ (h^{-1})_{IJ} & \tau_1(h^{-1}\boldsymbol{\tau})_I^J \end{pmatrix} \begin{pmatrix} \alpha^J \\ \beta_J \end{pmatrix} \quad (\text{B.5})$$

where

$$h^{IJ} = \int_{M_4} \omega^I * \omega^J \quad \text{and} \quad \boldsymbol{\tau}^{IJ} = \int_{M_4} \omega^I \wedge \omega^J.$$

Now choose the basis ζ^I of type $(1, 0)$. By definition ζ^I is a basis of solutions of the equation $J \cdot \zeta^I = i\zeta^I$. One can express the complex structure J in terms of the components of the complex period matrix \mathbf{T} . To this end we choose a basis ζ^I of the form

$$\zeta^I = \alpha^I + \mathbf{T}^{IJ} \beta_J. \quad (\text{B.6})$$

From $g(\zeta^I, \zeta^J) = g(\zeta^J, \zeta^I)$ we learn that \mathbf{T}^{IJ} is symmetric, and g is of type $(1, 1)$. Note that $g(\zeta^I, \bar{\zeta}^J) = 2 \text{Im } \mathbf{T}^{IJ}$. From Eq. (B.5) one finds

$$\mathbf{T}^{IJ} = \tau_1 \boldsymbol{\tau}^{IJ} + i\tau_2 h^{IJ}. \quad (\text{B.7})$$

We write $\mathbf{T} = X + iY$ and

$$J \cdot \begin{pmatrix} \alpha^I \\ \beta_I \end{pmatrix} = \begin{pmatrix} -(XY^{-1})^I_J & -(Y + XY^{-1}X)^{IJ} \\ (Y^{-1})_{IJ} & (Y^{-1}X)_I^J \end{pmatrix} \begin{pmatrix} \alpha^J \\ \beta_J \end{pmatrix} \quad (\text{B.8})$$

Let $\nu = n_I \alpha^I + m_I \beta^I$ and $\tilde{\nu} = \tilde{n}_I \alpha^I + \tilde{m}_I \beta_I$, then the metric in the α, β basis is:

$$g(\nu, \tilde{\nu}) = \begin{pmatrix} n_I & m_I \end{pmatrix} \begin{pmatrix} (Y + XY^{-1}X)^{IJ} & -(XY^{-1})^I_J \\ -(Y^{-1}X)_I^J & (Y^{-1})_{IJ} \end{pmatrix} \begin{pmatrix} \tilde{n}_J \\ \tilde{m}^J \end{pmatrix}. \quad (\text{B.9})$$

B.2 Splitting the instanton sum

Define level k theta functions to be

$$\Theta_{\beta, k}(\xi, \mathbf{T}) = \sum_{s_I \in \mathbb{Z}} \exp \left[2\pi i k \left(s_I + \frac{1}{2k} \beta_I \right) \mathbf{T}^{IJ} \left(s_J + \frac{1}{2k} \beta_J \right) \right] e^{2\pi i \xi^I (2k s_I + \beta_I)}.$$

Here \mathbf{T}^{IJ} is the complex period matrix (B.7). We can rewrite this expression in a geometrical form. Define,

$$s = s_I \omega^I, \quad \beta = \beta_I \omega^I, \quad \xi = \xi^I \hat{\omega}_I.$$

Then the Θ -function is

$$\Theta_{\beta,k}(\xi; \tau, *) = \sum_{s \in \text{Harm}_{\mathbb{Z}}^2(M_4)} e^{2\pi i k \tau \int_{M_4} (s + \frac{1}{2k} \beta)_+^2 + 2\pi i k \bar{\tau} \int_{M_4} (s + \frac{1}{2k} \beta)_-^2 + 2\pi i \int_{M_4} \xi \wedge (2ks + \beta)}. \quad (\text{B.10})$$

We want to express

$$S := \sum_{\nu \in V_{\mathbb{Z}}} \varphi(\nu) e^{-\frac{1}{2} \pi N g(\nu, \nu) + \Omega(\nu, \tilde{\ell})} \quad (\text{B.11})$$

where g is defined in (B.9) in terms of theta functions for the complex torus $V_{\mathbb{R}}/V_{\mathbb{Z}}$. Here $\nu = n_I \alpha^I + m^I \beta_I$, $\tilde{\ell} = -\ell_I^2 \alpha^I + \ell_1^I \beta_I$, therefore $\Omega(\nu, \tilde{\ell}) = n_I \ell_1^I + m^I \ell_I^2$. $\varphi(\nu)$ is a quadratic refinement of Ω , i.e.

$$\varphi(\nu_1 + \nu_2) = \varphi(\nu_1) \varphi(\nu_2) e^{i\pi N \Omega(\nu_1, \nu_2)}.$$

For our purpose it sufficient to consider $\varphi(\nu)$ of the form $\varphi(\nu) = e^{+i\pi N n_I m^I}$. Geometrically the sum (B.11) is

$$S = \sum_{\omega_c, \omega_b \in \text{Harm}_{\mathbb{Z}}^2(M_4)} e^{-\frac{\pi N}{2\tau_2} \int_{M_4} (\omega_c - \bar{\tau} \omega_b) * (\omega_c - \tau \omega_b) + i\pi N \int_{M_4} \omega_c \wedge \omega_b} e^{+ \int_{M_4} \omega_b \wedge \ell_1 + \omega_c \wedge \ell^2}. \quad (\text{B.12})$$

Now we do Poisson resummation over m^I :

$$S(\ell) = \left(\frac{2\tau_2}{N}\right)^{b_2/2} e^{\frac{1}{2\pi N} \ell_I^2 Y^{IJ} \ell_J^2} \sum_{n_I, w_I} e^{i\pi N (p_L)_I \mathbf{T}^{IJ} (p_L)_J - i\pi N (p_R)_I \bar{\mathbf{T}}^{IJ} (p_R)_J + (p_L)_I \psi^I + (p_R)_I \bar{\psi}^I}. \quad (\text{B.13})$$

Here

$$(p_L)_I = \frac{1}{2} n_I + \frac{1}{N} \left(w_I + \frac{N}{2} n_I\right); \quad (p_R)_I = \frac{1}{2} n_I - \frac{1}{N} \left(w_I + \frac{N}{2} n_I\right); \quad \psi^I = \ell_1^I + T^{IJ} \ell_J^2. \quad (\text{B.14})$$

Now we write $w_I = \beta_I - N s_I$ where $\beta_I \in \{0, 1, \dots, N-1\}$ and $s_I \in \mathbb{Z}$. In this case

$$(p_L)_I = n_I - s_I + \frac{1}{N} \beta_I \quad \text{and} \quad (p_R)_I = s_I - \frac{1}{N} \beta_I. \quad (\text{B.15})$$

Finally the sum (B.13) takes the form

$$S(\ell) = \left(\frac{2\tau_2}{N}\right)^{b_2/2} e^{\frac{1}{2\pi N} \ell_I^2 Y^{IJ} \ell_J^2} \sum_{\beta \in (\mathbb{Z}/N\mathbb{Z})^{b_2}} \Theta_{\beta, N/2} \left(\frac{1}{2\pi i N} \psi^I, \mathbf{T}^{IJ}\right) \Theta_{-\beta, N/2} \left(\frac{1}{2\pi i N} \bar{\psi}^I, -\bar{\mathbf{T}}^{IJ}\right). \quad (\text{B.16})$$

B.3 Summary

The final result is that the Gaussian sum

$$S = \sum_{\omega_c, \omega_b \in \Lambda} e^{-\frac{\pi N}{2\tau_2} \int_{M_4} (\omega_c - \bar{\tau}\omega_b) * (\omega_c - \tau\omega_b) + i\pi N \int_{M_4} \omega_c \wedge \omega_b} \times e^{\frac{1}{2i\tau_2} \int_{M_4} [\psi_+ \wedge (\omega_c - \bar{\tau}\omega_b)_+ - \psi_- \wedge (\omega_c - \tau\omega_b)_- - \bar{\psi}_+ \wedge (\omega_c - \tau\omega_b)_+ + \bar{\psi}_- \wedge (\omega_c - \bar{\tau}\omega_b)_-]} \quad (\text{B.17})$$

where $\Lambda = \text{Harm}_{\mathbb{Z}}^2(M_4)$ can be written as

$$S = \left(\frac{2\tau_2}{N}\right)^{b_2/2} e^{-\frac{1}{8\pi N\tau_2} \int_{M_4} (\psi - \bar{\psi}) * (\psi - \bar{\psi})} \sum_{\beta \in \Lambda/\Lambda_N} \Theta_{\beta, N/2} \left(\frac{1}{2\pi i N} \psi; \tau, * \right) \Theta_{-\beta, N/2} \left(\frac{1}{2\pi i N} \bar{\psi}; -\bar{\tau}, * \right). \quad (\text{B.18})$$

where $\Lambda_N = \text{Harm}_{N\mathbb{Z}}^2(M_4)$.

For the sum (4.15) we have

$$\begin{aligned} \psi_+ &= -2\pi i N (c - \tau b)_+, & \psi_- &= -2\pi i N (c - \bar{\tau} b)_-; \\ \bar{\psi}_+ &= 2\pi i (v_c - \bar{\tau} v_b)_+, & \bar{\psi}_- &= 2\pi i (v_c - \tau v_b)_-. \end{aligned}$$

For these specific values of ψ and $\bar{\psi}$ the sum (B.18) can be rewritten in terms of the Siegel-Narain Θ -function defined by Eq. (4.17). It has the following modular properties ($\gamma \in \frac{1}{2k}\Lambda$):

$$T : \quad \Theta_{\Lambda+\gamma, k}(\tau+1, c+b+\frac{1}{2}w_2, b) = e^{2\pi i k(\gamma, \gamma) - 2\pi i k(w_2, \gamma) - i\pi k(w_2, b)} \Theta_{\Lambda+\gamma, k}(\tau, c, b); \quad (\text{B.19a})$$

$$S : \quad \Theta_{\Lambda+\gamma, k}(-1/\tau, b, -c) = \frac{(-i\tau)^{b_2^+/2} (i\bar{\tau})^{b_2^-/2}}{(2k)^{b_2/2}} \sum_{\gamma' \in (\frac{1}{2k}\Lambda)/\Lambda} e^{-4\pi i k(\gamma, \gamma')} \Theta_{\Lambda+\gamma', k}(\tau, c, b). \quad (\text{B.19b})$$

Here w_2 is a characteristic vector, such that

$$(\omega, \omega) = (\omega, w_2) \pmod{2}.$$

For a spin manifold it equals zero.

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